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THE FIGURED 12 SUBCASES OF THE COEFFICIENTS INTERRELATIONSHIPS IN THE KERNEL OF A CONTINUOUS STRICTLY CONVEX ANTAGONISTIC GAME WITH THE CORRESPONDING EIGHT TYPES OF THE SOLUTION

There have been investigated 12 subcases of some coefficients and some their sums signs in the kernel of a continuous strictly convex antagonistic game. From that the eight game solution types have been determined, that had been displayed in the conclusion table.

Исследовано 12 подслучаев знаков некоторых коэффициентов и некоторых их сумм в ядре одной непрерывной строго выпуклой антагонистической игры. Из этого определено восемь типов решения игры, которые отображены в итоговой таблице.

The issue specification in the paper and the investigation goal designation

Without arguing, that there are many conflicting events and processes are structured, formalized and investigated with the mathematical gaming and simulation modeling. Antagonistic games give the exclusively relevant and good fitting mathematical model for making decisions in some competitive activity economical processes, where the two players are the rivals. Then an actual investigation goal designation in the antagonistic games lies in finding all the solutions

$$\mathscr{S} = \left\{ \mathscr{X}_{\text{opt}},\ \mathscr{Y}_{\text{opt}},\ V_{\text{opt}} \right\} \tag{1}$$

of the convex continuous antagonistic games [1], which kernel $S(x,\ y)$ as the surface is defined generally on the unit square

$$D_S = X \times Y = [0;\ 1] \times [0;\ 1], \tag{2}$$

where $x \in X = [0;\ 1]$ and $y \in Y = [0;\ 1]$ are the pure strategies of the first and second players respectively. There in the formula (1) the denomination V_{opt} is assigned as the game value. The optimal strategies set of the first player is $\mathscr{X}_{\text{opt}}$, and the optimal strategies set of the second player is $\mathscr{Y}_{\text{opt}}$. This paper investigation goal designation is to find all the solutions (1) of the continuous strictly convex antagonistic game with the kernel

$$S(x,\ y) = ax^2 + bx + gxy + cy + hy^2 + k\ , \tag{3}$$

which is defined on the unit square D_S, where $a > 0$, $b > 0$, $g < 0$, $c < 0$, $k \in \mathbb{R}$. As this game is said to be the strictly convex, then $\forall\ x \in X$ and $\forall\ y \in Y$ there must be held the condition $\frac{\partial^2 S(x,\ y)}{\partial y^2} > 0$, whence $\frac{\partial^2 S(x,\ y)}{\partial y^2} = 2h > 0$ and the coefficient $h > 0$. While solving this game there should be applied the known maximin method [2, 3 — 8] with the total determining the sets $\mathscr{X}_{\text{opt}}$ and $\mathscr{Y}_{\text{opt}}$ [9 — 13].

The total solving of the specified continuous strictly convex antagonistic game

First of all mark, that as $a > 0$ then the parabola (3) being the function of the only variable x does not have the global maximum point. Then this parabola on the unit segment $X = [0;\ 1]$ reaches the maximum either in the point $x = 0$ or $x = 1$ and, certainly, this maximum depends upon the sign of the statement $a + b + gy$. While having $a > 0$, $b > 0$, $g < 0$, $c < 0$ then there $a + b + gy > 0$ if $y < -\frac{a+b}{g}$. The value $-\frac{a+b}{g} > 0$ and $-\frac{a+b}{g} < 1$ if $a + b + g < 0$.

Subcase 1. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g < 0$. Here $-\frac{a+b}{g} < 1$ and as $a + b + gy > 0$ $\forall\ y < -\frac{a+b}{g}$

by $-\frac{a+b}{g} \in (0;\,1) \subset Y$ then the maximum of the surface (3) on the unit segment X of the variable x is

$$\max_{x \in X} S(x,\, y) = \begin{cases} S(1,\, y) = a + b + gy + cy + hy^2 + k,\; y \in \left[0;\, -\frac{a+b}{g}\right], \\ S(0,\, y) = cy + hy^2 + k,\; y \in \left[-\frac{a+b}{g};\, 1\right]. \end{cases} \quad (4)$$

Before finding the local minimum of the parabola $S(1,\, y)$ on some subsegment of the unit segment Y primarily the global minimum of $S(1,\, y)$ should be determined. The first derivative of the parabola $S(1,\, y)$ is

$$\frac{d}{dy} S(1,\, y) = \frac{d}{dy}\left(a + b + gy + cy + hy^2 + k\right) = g + c + 2hy\,. \quad (5)$$

The first critical point of the parabola $S(1,\, y)$ is the zero point of the line (5), that is $y = y_{\text{cr}}^{\langle 1 \rangle} = -\frac{g+c}{2h}$, and as the second derivative of the parabola $S(1,\, y)$ is

$$\frac{d^2}{dy^2} S(1,\, y) = \frac{d}{dy}\left(g + c + 2hy\right) = 2h > 0\,, \quad (6)$$

then the global minimum of the parabola $S(1,\, y)$ is $y_{\text{cr}}^{\langle 1 \rangle} = y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h}$. Analogously to that there should be determined the global minimum of the parabola $S(0,\, y)$. The first derivative of the parabola $S(0,\, y)$ is

$$\frac{d}{dy} S(0,\, y) = \frac{d}{dy}\left(cy + hy^2 + k\right) = c + 2hy \quad (7)$$

and the first critical point of the parabola $S(0,\, y)$ is the zero point of the line (7) $y = y_{\text{cr}}^{\langle 0 \rangle} = -\frac{c}{2h}$. The second derivative of the parabola $S(0,\, y)$ is the same as the second derivative of the parabola $S(1,\, y)$, that is

$$\frac{d^2}{dy^2} S(0,\, y) = \frac{d}{dy}\left(c + 2hy\right) = 2h > 0 \quad (8)$$

and the global minimum of the parabola $S(0,\, y)$ is $y_{\text{cr}}^{\langle 0 \rangle} = y_{\min}^{\langle 0 \rangle} = -\frac{c}{2h}$.

Further will determine whether $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in \left[0;\, -\frac{a+b}{g}\right]$ or $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \notin \left[0;\, -\frac{a+b}{g}\right]$. As $-\frac{g+c}{2h} > 0$ then $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in (0;\, 1]$ by $-\frac{g+c}{2h} \leqslant 1$. That is $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in (0;\, 1]$ by $g + c + 2h \geqslant 0$ and $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} > 1$ by $g + c + 2h < 0$.

Subcase 1.1. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g < 0$; $g + c + 2h < 0$. As the point $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} > 1$ then here is the triple parabolic inequality

$$S(1,\, 0) > S\left(1,\, -\frac{a+b}{g}\right) > S(1,\, 1) > S\left(1,\, -\frac{g+c}{2h}\right) = S\left(1,\, y_{\min}^{\langle 1 \rangle}\right). \quad (9)$$

The point $y_{\min}^{\langle 0 \rangle} = -\frac{c}{2h} > 0$ and as

$$-\frac{c}{2h}-\left(-\frac{a+b}{g}\right)=-\frac{c}{2h}+\frac{a+b}{g}=\frac{2h(a+b)-cg}{2hg} \tag{10}$$

then $y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}>-\frac{a+b}{g}$ by $2h(a+b)-cg<0$. For furtherance also note that

$$S\left(1,\,-\frac{a+b}{g}\right)=S\left(0,\,-\frac{a+b}{g}\right). \tag{11}$$

Subcase 1.1.1. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h<0$; $2h(a+b)-cg<0$. The difference

$$y_{\min}^{\langle 0\rangle}-y_{\min}^{\langle 1\rangle}=-\frac{c}{2h}-\left(-\frac{g+c}{2h}\right)=-\frac{c}{2h}+\frac{g+c}{2h}=\frac{g}{2h}<0 \tag{12}$$

shows that $y_{\min}^{\langle 0\rangle}<y_{\min}^{\langle 1\rangle}$. So there may be as $y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}\in\left(-\frac{a+b}{g};\,1\right]$ by $c+2h\geqslant 0$ as well as $y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}>1$ by $c+2h<0$.

Subcase 1.1.1.1. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h<0$; $2h(a+b)-cg<0$; $c+2h\geqslant 0$. As the point $y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}\in\left(-\frac{a+b}{g};\,1\right]$ then

$$S\left(0,\,-\frac{a+b}{g}\right)>S\left(0,\,-\frac{c}{2h}\right)=S\left(0,\,y_{\min}^{\langle 0\rangle}\right), \tag{13}$$

and with (11) and the triple parabolic inequality (9) the minimum of the function (4)

$$\begin{aligned}
\min_{y\in Y}\max_{x\in X} S(x,\,y)&=\min\left\{\min_{y\in\left[0;\,-\frac{a+b}{g}\right]} S(1,\,y),\ \min_{y\in\left[-\frac{a+b}{g};\,1\right]} S(0,\,y)\right\}=\\
=\min\left\{\min\left\{S(1,\,0),\,S\left(1,\,-\frac{a+b}{g}\right)\right\},\,S\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}&=\min\left\{\min\left\{S(1,\,0),\,S\left(1,\,-\frac{a+b}{g}\right)\right\},\,S\left(0,\,-\frac{c}{2h}\right)\right\}=\\
=\min\left\{S\left(1,\,-\frac{a+b}{g}\right),\,S\left(0,\,-\frac{c}{2h}\right)\right\}&=S\left(0,\,-\frac{c}{2h}\right)=S\left(0,\,y_{\min}^{\langle 0\rangle}\right)=\\
=c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k&=k-\frac{c^2}{4h}=V_{\mathrm{opt}}
\end{aligned} \tag{14}$$

is reached in the point $y=y_{\mathrm{opt}}=-\frac{c}{2h}$, that is on the set of the second player optimal strategies

$$\mathscr{Y}_{\mathrm{opt}}=Y_{\mathrm{opt}}=\left\{-\frac{c}{2h}\right\}=\left\{y_{\mathrm{opt}}\right\}, \tag{15}$$

which coincides with the second player optimal pure strategies set Y_{opt}. The set of the first player optimal pure strategies X_{opt} primarily should be determined by the roots x_1 and x_2 of the quadratic equation [2, 13]

$$V_{\mathrm{opt}}=S\left(x,\,y_{\mathrm{opt}}\right). \tag{16}$$

Hereon the corresponding equation (16) is

$$V_{\mathrm{opt}}=S\left(0,-\frac{c}{2h}\right)=k-\frac{c^2}{4h}=ax^2+bx+gx\left(-\frac{c}{2h}\right)+c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k=$$
$$=ax^2+bx+gx\left(-\frac{c}{2h}\right)+k-\frac{c^2}{4h}=x\left(ax+b-\frac{cg}{2h}\right)+k-\frac{c^2}{4h}=$$
$$=x\left(ax+\frac{2hb-cg}{2h}\right)+k-\frac{c^2}{4h}=S\left(x,-\frac{c}{2h}\right)=S\left(x,\,y_{\mathrm{opt}}\right). \tag{17}$$

From the equation (17) get the equation

$$x\left(ax+\frac{2hb-cg}{2h}\right)=x\left(x+\frac{2hb-cg}{2ah}\right)=0, \tag{18}$$

where the roots of the equation (16) are $x_1=0$ and $x_2=\frac{cg-2hb}{2ah}$. But the initial condition $2h(a+b)-cg<0$ means that $x_2=\frac{cg-2hb}{2ah}>1$. So here $x_1\in X$ and $x_2\notin X$. Thereupon the set

$$X_{\mathrm{opt}}=\{x_1\}=\{0\}=\mathscr{X}_{\mathrm{opt}} \tag{19}$$

and the investigated subcase game solution is the set

$$\mathscr{S}=\left\{\{0\},\left\{-\frac{c}{2h}\right\},\,k-\frac{c^2}{4h}\right\}. \tag{20}$$

Subcase 1.1.1.2. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h<0$; $2h(a+b)-cg<0$; $c+2h<0$. As the point $y_{\min}^{\langle 0\rangle}=-\frac{c}{2h}>1$ then here is the triple parabolic inequality

$$S(0,0)>S\left(0,-\frac{a+b}{g}\right)>S(0,1)>S\left(0,-\frac{c}{2h}\right)=S\left(0,\,y_{\min}^{\langle 0\rangle}\right). \tag{21}$$

The inequalities (9) and (21) with (11) drive to that the minimum of the function (4)

$$\min_{y\in Y}\max_{x\in X}S(x,y)=\min\left\{\min_{y\in\left[0;\,-\frac{a+b}{g}\right]}S(1,y),\ \min_{y\in\left[-\frac{a+b}{g};\,1\right]}S(0,y)\right\}=$$
$$=\min\left\{\min\left\{S(1,0),\,S\left(1,-\frac{a+b}{g}\right)\right\},\,\min\left\{S\left(0,-\frac{a+b}{g}\right),\,S(0,1)\right\}\right\}=\min\left\{S\left(1,-\frac{a+b}{g}\right),\,S(0,1)\right\}=$$
$$=\min\left\{S\left(0,-\frac{a+b}{g}\right),\,S(0,1)\right\}=S(0,1)=c+h+k=V_{\mathrm{opt}} \tag{22}$$

is reached on the set

$$\mathscr{Y}_{\mathrm{opt}}=Y_{\mathrm{opt}}=\{1\}=\left\{y_{\mathrm{opt}}\right\}. \tag{23}$$

The roots of the corresponding equation (16)

$$V_{\mathrm{opt}}=S(0,1)=c+h+k=ax^2+bx+gx+c+h+k=$$
$$=ax\left(x+\frac{b+g}{a}\right)+c+h+k=S(x,1)=S\left(x,\,y_{\mathrm{opt}}\right) \tag{24}$$

are $x_1=0$ and $x_2=-\frac{b+g}{a}$. But $a+b+g<0$ means $-(b+g)>a>0$ and $-\frac{b+g}{a}>1$. Then $x_1\in X$, $x_2\notin X$ and there is the set (19), whence the investigated subcase game solution is

$$\mathscr{S} = \left\{ \{0\},\, \{1\},\, c+h+k \right\}. \tag{25}$$

Subcase 1.1.2. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h<0$; $2h(a+b)-cg \geqslant 0$. Here the point $y_{\min}^{\langle 0 \rangle} = -\dfrac{c}{2h} \leqslant -\dfrac{a+b}{g}$ and there is the true double parabolic inequality

$$S\left(0,\, y_{\min}^{\langle 0 \rangle}\right) = S\left(0,\, -\frac{c}{2h}\right) \leqslant S\left(0,\, -\frac{a+b}{g}\right) < S(0,\, 1). \tag{26}$$

This inequality with the inequality (9) and equality (11) give that the minimum of the function (4)

$$\min_{y \in Y} \max_{x \in X} S(x,\, y) = \min\left\{ \min_{y \in \left[0;\, -\frac{a+b}{g}\right]} S(1,\, y),\ \min_{y \in \left[-\frac{a+b}{g};\, 1\right]} S(0,\, y) \right\} =$$
$$= \min\left\{ \min\left\{ S(1,\, 0),\, S\left(1,\, -\frac{a+b}{g}\right) \right\},\ \min\left\{ S\left(0,\, -\frac{a+b}{g}\right),\, S(0,\, 1) \right\} \right\} =$$
$$= \min\left\{ S\left(1,\, -\frac{a+b}{g}\right),\, S\left(0,\, -\frac{a+b}{g}\right) \right\} = S\left(1,\, -\frac{a+b}{g}\right) = S\left(0,\, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = V_{\text{opt}} \tag{27}$$

is reached on the set

$$\mathscr{Y}_{\text{opt}} = Y_{\text{opt}} = \left\{ -\frac{a+b}{g} \right\} = \left\{ y_{\text{opt}} \right\}. \tag{28}$$

The roots of the corresponding equation (16)

$$V_{\text{opt}} = S\left(1,\, -\frac{a+b}{g}\right) = S\left(0,\, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = ax^2 + bx + gx\left(-\frac{a+b}{g}\right) + c\left(-\frac{a+b}{g}\right) + h\left(-\frac{a+b}{g}\right)^2 + k =$$
$$= ax^2 - ax + h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = ax(x-1) + h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k = S\left(x,\, -\frac{a+b}{g}\right) = S\left(x,\, y_{\text{opt}}\right) \tag{29}$$

are $x_1 = 0$ and $x_2 = 1$. They are such that $x_1 \in X$ and $x_2 \in X$, so the set

$$X_{\text{opt}} = \{x_1,\, x_2\} = \{0,\, 1\}. \tag{30}$$

May $P(x_1)$ and $P(x_2)$ be the probabilities of the first player selecting its pure strategies $x_1 = x_{\text{opt}}^{\langle 1 \rangle}$ and $x_2 = x_{\text{opt}}^{\langle 2 \rangle}$. Then the set $\mathscr{X}_{\text{opt}} = \left\{ X_{\text{opt}},\, \left\{ P\left(x_{\text{opt}}^{\langle 1 \rangle}\right),\, P\left(x_{\text{opt}}^{\langle 2 \rangle}\right) \right\} \right\}$ and there are $X_{\text{opt}} = \left\{ x_{\text{opt}}^{\langle 1 \rangle},\, x_{\text{opt}}^{\langle 2 \rangle} \right\}$, $P\left(x_{\text{opt}}^{\langle 1 \rangle}\right) + P\left(x_{\text{opt}}^{\langle 2 \rangle}\right) = 1$. Those probabilities satisfy the double inequality [1, 9]

$$S\left(x^{\langle 1 \rangle},\, y_{\text{opt}}\right) P\left(x^{\langle 1 \rangle}\right) + S\left(x^{\langle 2 \rangle},\, y_{\text{opt}}\right) P\left(x^{\langle 2 \rangle}\right) \leqslant V_{\text{opt}} \leqslant S\left(x_{\text{opt}}^{\langle 1 \rangle},\, y\right) P\left(x_{\text{opt}}^{\langle 1 \rangle}\right) + S\left(x_{\text{opt}}^{\langle 2 \rangle},\, y\right) P\left(x_{\text{opt}}^{\langle 2 \rangle}\right), \tag{31}$$

where $y \neq y_{\text{opt}}$, and $x^{\langle 1 \rangle} \neq x_{\text{opt}}^{\langle 1 \rangle}$, or $x^{\langle 2 \rangle} \neq x_{\text{opt}}^{\langle 2 \rangle}$, or $P\left(x^{\langle 1 \rangle}\right) \neq P\left(x_{\text{opt}}^{\langle 1 \rangle}\right)$. In the being investigated subcase the probabilities $P(x_1) = P(0)$ and $P(x_2) = P(1)$ may be determined from the right inequality (31):

$$V_{\text{opt}} = S\left(0,\, -\frac{a+b}{g}\right) = S\left(1,\, -\frac{a+b}{g}\right) = h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k \leqslant S(0,\, y)P(0) + S(1,\, y)P(1) =$$
$$= \left(cy + hy^2 + k\right)P(0) + \left(a + b + gy + cy + hy^2 + k\right)P(1) = cy + hy^2 + (a + b + gy)P(1) + k; \tag{32}$$

$$h\frac{(a+b)^2}{g^2}-c\frac{a+b}{g}-cy-hy^2=h\frac{(a+b)^2-g^2y^2}{g^2}-c\frac{a+b+gy}{g}=$$

$$=h\frac{(a+b+gy)(a+b-gy)}{g^2}-c\frac{a+b+gy}{g}=(a+b+gy)\left[\frac{h(a+b-gy)-cg}{g^2}\right]\leqslant(a+b+gy)P(1). \quad (33)$$

While $a+b+gy>0$ then $y<-\frac{a+b}{g}$ and step by step $-gy<a+b$, $a+b-gy<2(a+b)$,

$$\frac{h(a+b-gy)-cg}{g^2}<\frac{2h(a+b)-cg}{g^2}. \quad (34)$$

But $-\frac{a+b}{g}<-\frac{g+c}{2h}$ and here is their difference

$$-\frac{g+c}{2h}-\left(-\frac{a+b}{g}\right)=-\frac{g+c}{2h}+\frac{a+b}{g}=\frac{2h(a+b)-g(g+c)}{2hg}, \quad (35)$$

where $\frac{2h(a+b)-g(g+c)}{2hg}>0$. This gives that $2h(a+b)-g(g+c)<0$ and then goes the corollary

$$\frac{2h(a+b)-cg}{g^2}\in[0;1). \quad (36)$$

And then from the statement (33) there is an inequality for the probability $P(1)$ while $a+b+gy>0$:

$$\frac{h(a+b-gy)-cg}{g^2}\leqslant P(1)\in\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\left[\frac{2h(a+b)-cg}{g^2}-\varepsilon;1\right]. \quad (37)$$

While $a+b+gy<0$ then $y>-\frac{a+b}{g}$ and again $-gy>a+b$, $a+b-gy>2(a+b)$,

$$\frac{h(a+b-gy)-cg}{g^2}>\frac{2h(a+b)-cg}{g^2}, \quad (38)$$

whence from the statement (33) there is an inequality for the probability $P(1)$ while $a+b+gy<0$:

$$\frac{h(a+b-gy)-cg}{g^2}\geqslant P(1)\in\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\left[0;\frac{2h(a+b)-cg}{g^2}+\varepsilon\right]. \quad (39)$$

Therefore the probability $P(1)$ is the intersection of the segments in the formulas (37) and (39):

$$P(1)\in\lim_{\substack{\varepsilon>0\\ \varepsilon\to0}}\left\{\left[0;\frac{2h(a+b)-cg}{g^2}+\varepsilon\right]\cap\left[\frac{2h(a+b)-cg}{g^2}-\varepsilon;1\right]\right\}=\left\{\frac{2h(a+b)-cg}{g^2}\right\}. \quad (40)$$

Hence the probability of the first player selecting its pure strategy $x_1=0$ is

$$P(0)=1-P(1)=1-\frac{2h(a+b)-cg}{g^2}=\frac{g(g+c)-2h(a+b)}{g^2}. \quad (41)$$

Finally in the investigated subcase the set

$$\mathscr{X}_{\text{opt}} = \left\{ \{0,\,1\},\, \left\{ \frac{g(g+c)-2h(a+b)}{g^2},\, \frac{2h(a+b)-cg}{g^2} \right\} \right\} \tag{42}$$

and the game solution

$$\mathscr{S} = \left\{ \left\{ \{0,\,1\},\, \left\{ \frac{g(g+c)-2h(a+b)}{g^2},\, \frac{2h(a+b)-cg}{g^2} \right\} \right\},\, \left\{ -\frac{a+b}{g} \right\},\, h\frac{(a+b)^2}{g^2} - c\frac{a+b}{g} + k \right\}. \tag{43}$$

Subcase 1.2. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h \geqslant 0$. The point $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in (0;\,1]$ and with the difference (35) there are $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in \left(0;\, -\dfrac{a+b}{g}\right)$ by $2h(a+b)-g(g+c)>0$ and $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \geqslant -\dfrac{a+b}{g}$ by $2h(a+b)-g(g+c) \leqslant 0$.

Subcase 1.2.1. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g<0$; $g+c+2h \geqslant 0$; $2h(a+b)-g(g+c)>0$. As the point $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in \left(0;\, -\dfrac{a+b}{g}\right)$ then $y_{\min}^{\langle 0\rangle} < y_{\min}^{\langle 1\rangle}$ and $y_{\min}^{\langle 0\rangle} = -\dfrac{c}{2h} < -\dfrac{a+b}{g}$. As the corollary, here are the inequality

$$S\left(1,\, y_{\min}^{\langle 1\rangle}\right) = S\left(1,\, -\frac{g+c}{2h}\right) < S\left(1,\, -\frac{a+b}{g}\right) \tag{44}$$

and the double parabolic inequality

$$S\left(0,\, y_{\min}^{\langle 0\rangle}\right) = S\left(0,\, -\frac{c}{2h}\right) < S\left(0,\, -\frac{a+b}{g}\right) < S(0,\,1). \tag{45}$$

They with the equality (11) result in the conclusion, that the minimum of the function (4)

$$\min_{y\in Y}\max_{x\in X} S(x,\,y) = \min\left\{ \min_{y\in\left[0;\,-\frac{a+b}{g}\right]} S(1,\,y),\, \min_{y\in\left[-\frac{a+b}{g};\,1\right]} S(0,\,y) \right\} =$$

$$= \min\left\{ S\left(1,\, y_{\min}^{\langle 1\rangle}\right),\, \min\left\{ S\left(0,\, -\frac{a+b}{g}\right),\, S(0,\,1) \right\} \right\} = \min\left\{ S\left(1,\, -\frac{g+c}{2h}\right),\, \min\left\{ S\left(0,\, -\frac{a+b}{g}\right),\, S(0,\,1) \right\} \right\} =$$

$$= \min\left\{ S\left(1,\, -\frac{g+c}{2h}\right),\, S\left(0,\, -\frac{a+b}{g}\right) \right\} = S\left(1,\, -\frac{g+c}{2h}\right) =$$

$$= a+b+g\left(-\frac{g+c}{2h}\right) + c\left(-\frac{g+c}{2h}\right) + h\left(-\frac{g+c}{2h}\right)^2 + k =$$

$$= a+b+(g+c)\left(-\frac{g+c}{2h}\right) + h\left(-\frac{g+c}{2h}\right)^2 + k = a+b-\frac{(g+c)^2}{2h} + \frac{(g+c)^2}{4h} + k = a+b-\frac{(g+c)^2}{4h} + k = V_{\text{opt}} \tag{46}$$

is reached on the set

$$\mathscr{Y}_{\text{opt}} = Y_{\text{opt}} = \left\{ -\frac{g+c}{2h} \right\} = \{ y_{\text{opt}} \}. \tag{47}$$

The corresponding equation (16) is

$$V_{\text{opt}} = S\left(1,\, -\frac{g+c}{2h}\right) = a+b-\frac{(g+c)^2}{4h} + k = ax^2 + bx + gx\left(-\frac{g+c}{2h}\right) + c\left(-\frac{g+c}{2h}\right) + h\left(-\frac{g+c}{2h}\right)^2 + k =$$

$$= ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} + k = S\left(x, -\frac{g+c}{2h}\right) = S\left(x, y_{\text{opt}}\right); \tag{48}$$

$$ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} - a - b + \frac{(g+c)^2}{4h} = ax^2 + x\left[\frac{2bh - g(g+c)}{2h}\right] + \frac{g(g+c) - 2h(a+b)}{2h} =$$

$$= a\left(x^2 + x\left[\frac{2bh - g(g+c)}{2ah}\right] + \frac{g(g+c) - 2h(a+b)}{2ah}\right) = a(x-1)\left(x - \frac{g(g+c) - 2h(a+b)}{2ah}\right) = 0. \tag{49}$$

As it is seen from the equation (49), the roots of the corresponding equation (48) are $x_1 = \frac{g(g+c) - 2h(a+b)}{2ah}$ and $x_2 = 1$. But for $2h(a+b) - g(g+c) > 0$ the root $x_1 = \frac{g(g+c) - 2h(a+b)}{2ah} < 0$ and here $x_1 \notin X$, $x_2 \in X$. Thereupon the set

$$X_{\text{opt}} = \{x_2\} = \{1\} = \mathscr{X}_{\text{opt}} \tag{50}$$

and the investigated subcase game solution is the set

$$\mathscr{S} = \left\{\{1\}, \left\{-\frac{g+c}{2h}\right\}, a + b - \frac{(g+c)^2}{4h} + k\right\}. \tag{51}$$

Subcase 1.2.2. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g < 0$; $g + c + 2h \geqslant 0$; $2h(a+b) - g(g+c) \leqslant 0$. The point $y_{\min}^{\langle 1\rangle} = -\frac{g+c}{2h} \geqslant -\frac{a+b}{g}$, but else there is the need for learning the point $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h}$ position. Knowing that $g + c + 2h \geqslant 0$ and $g < 0$, the associated statement $c + 2h > 0$ gives the condition $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} < 1$. Then from the difference (10) have that $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} \leqslant -\frac{a+b}{g}$ by $2h(a+b) - cg \geqslant 0$ and $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} > -\frac{a+b}{g}$ by $2h(a+b) - cg < 0$.

Subcase 1.2.2.1. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g < 0$; $g + c + 2h \geqslant 0$; $2h(a+b) - g(g+c) \leqslant 0$; $2h(a+b) - cg \geqslant 0$. As the point $y_{\min}^{\langle 1\rangle} = -\frac{g+c}{2h} \geqslant -\frac{a+b}{g}$ then there is the double parabolic inequality

$$S(1, 0) > S\left(1, -\frac{a+b}{g}\right) \geqslant S\left(1, -\frac{g+c}{2h}\right) = S\left(1, y_{\min}^{\langle 1\rangle}\right). \tag{52}$$

And the point $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} \leqslant -\frac{a+b}{g}$ position drives to the double parabolic inequality (26). So, in this subcase the minimum of the function (4) is (27), being reached on the set (28). The roots of the corresponding equation (16) are the roots of the equation (29) and make the set (30). Further have the inequalities (37) and (39) for the probability $P(1)$ by either $a + b + gy > 0$ or $a + b + gy < 0$, where only just the value

$$\frac{2h(a+b) - cg}{g^2} \in [0; 1] = X \tag{53}$$

in the statement (40) for the probability $P(1)$. Thus the optimal strategies set of the first player is (42) and the investigated subcase game solution is the set (43).

Subcase 1.2.2.2. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g < 0$; $g + c + 2h \geqslant 0$; $2h(a+b) - g(g+c) \leqslant 0$; $2h(a+b) - cg < 0$. Here the point $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} > -\frac{a+b}{g}$ and $y_{\min}^{\langle 0\rangle} = -\frac{c}{2h} < 1$. Consequently, this point position is

$y_{\min}^{\langle 0\rangle} = -\dfrac{c}{2h} \in \left(-\dfrac{a+b}{g};\, 1\right)$, what drives to the inequality (13). The inequality (13) with the double parabolic inequality (52), minding the equality (11), drive to the conclusion, that the minimum of the function (4) is (14) and it is reached on the set (15). The corresponding equation (16) is the equation (17), which roots are $x_1 = 0$ and $x_2 = \dfrac{cg-2hb}{2ah}$. But the initial condition $2h(a+b)-cg<0$ means that $x_2 = \dfrac{cg-2hb}{2ah} > 1$. So here $x_1 \in X$ and $x_2 \notin X$. Thereupon the set (19) is true and the investigated subcase game solution is the set (20).

Subcase 2. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g>0$. Here is absolutely clear that $a+b+gy>0 \quad \forall\ y \in Y$. Therefore the maximum of the surface (3) on the unit segment X of the variable x is

$$\max_{x\in X} S(x,\, y) = \max\left\{S(0,\, y),\, S(1,\, y)\right\} = S(1,\, y) = a+b+gy+cy+hy^2+k\,. \tag{54}$$

The minimum of the parabola (54) depends upon whether $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in [0;\, 1]$ or $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \notin [0;\, 1]$, that is firstly upon the sign of the sum $g+c$. But yet $g+c<0$ and the point $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} > 0$. Then $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in (0;\, 1]$ by $g+c+2h \geqslant 0$ and $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} > 1$ by $g+c+2h<0$.

Subcase 2.1. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g>0$; $g+c+2h \geqslant 0$. As the point $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} \in (0;\, 1]$ then the minimum of the parabola (54)

$$\begin{aligned}\min_{y\in Y}\max_{x\in X} S(x,\, y) &= \min_{y\in Y} S(1,\, y) = S\left(1,\, y_{\min}^{\langle 1\rangle}\right) = S\left(1,\, -\frac{g+c}{2h}\right) = a+b+g\left(-\frac{g+c}{2h}\right)+c\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k = \\ &= a+b+(g+c)\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k = a+b-\frac{(g+c)^2}{2h}+\frac{(g+c)^2}{4h}+k = a+b-\frac{(g+c)^2}{4h}+k = V_{\text{opt}}\end{aligned} \tag{55}$$

is reached on the set (47). The roots of the corresponding equation (16) are the roots of the equation (48), which has been simplified to the equation (49). They are $x_1 = \dfrac{g(g+c)-2h(a+b)}{2ah}$ and $x_2 = 1$. But as $-\dfrac{a+b}{g} > 1$ then here also $-\dfrac{g+c}{2h} < -\dfrac{a+b}{g}$ and from the difference (35) the inequality $2h(a+b)-g(g+c)>0$ is true. Thus $\dfrac{g(g+c)-2h(a+b)}{2ah} < 0$ and $x_1 \notin X$, $x_2 \in X$, that gives the set (50), whence the investigated subcase game solution is (51).

Subcase 2.2. $a>0$, $b>0$, $g<0$, $c<0$; $a+b+g>0$; $g+c+2h<0$. As the point $y_{\min}^{\langle 1\rangle} = -\dfrac{g+c}{2h} > 1$ then there is the double parabolic inequality

$$S(1,\, 0) > S(1,\, 1) > S\left(1,\, -\frac{g+c}{2h}\right) = S\left(1,\, y_{\min}^{\langle 1\rangle}\right). \tag{56}$$

Thereupon the minimum of the parabola (54)

$$\min_{y\in Y}\max_{x\in X} S(x,\, y) = \min_{y\in Y} S(1,\, y) = \min\left\{S(1,\, 0),\, S(1,\, 1)\right\} = S(1,\, 1) = a+b+g+c+h+k = V_{\text{opt}} \tag{57}$$

is reached on the set (23). The roots of the corresponding equation (16)

$$\begin{aligned} V_{\text{opt}} &= S(1,\, 1) = a+b+g+c+h+k = ax^2+bx+gx+c+h+k = \\ &= a(x-1)\left(x+\frac{a+b+g}{a}\right)+a+b+g+c+h+k = S(x,\, 1) = S\left(x,\, y_{\text{opt}}\right)\end{aligned} \tag{58}$$

are $x_1 = -\frac{a+b+g}{a}$ and $x_2 = 1$. However, as $-\frac{a+b+g}{a} < 0$, then $x_1 \notin X$, $x_2 \in X$ and there is the set (50), whence the investigated subcase game solution is

$$\mathscr{S} = \left\{ \{1\}, \{1\}, a+b+g+c+h+k \right\}. \tag{59}$$

Subcase 3. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a+b+g = 0$. In this boundary subcase the maximum of the surface (3) on the unit segment X of the variable x is

$$\max_{x \in X} S(x,\, y) = \begin{cases} S(1,\, y) = a+b+gy+cy+hy^2+k,\; y \in [0;\, 1], \\ S(0,\, y) = S(1,\, y) = cy+hy^2+k,\; y \in \{1\}. \end{cases} \tag{60}$$

Apparently that the further minimization of the function (60) on the unit segment Y depends whether the point $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in [0;\, 1]$ or $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \notin [0;\, 1]$, that is firstly upon the sign of the sum $g+c$. But yet $g+c < 0$ and the point $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} > 0$, so then $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in (0;\, 1)$ by $g+c+2h > 0$, $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} > 1$ by $g+c+2h < 0$, and $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} = 1$ by $g+c+2h = 0$.

Subcase 3.1. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a+b+g = 0$; $g+c+2h > 0$. The point $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} \in [0;\, 1)$ and so the minimum of the function (60)

$$\begin{aligned} \min_{y \in Y} \max_{x \in X} S(x,\, y) &= \min\left\{ \min_{y \in [0;\, 1]} S(1,\, y),\, \min_{y \in \{1\}} S(0,\, y) \right\} = \min\left\{ S\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\, S(0,\, 1) \right\} = \min\left\{ S\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\, S(1,\, 1) \right\} = \\ &= S\left(1,\, y_{\min}^{\langle 1 \rangle}\right) = S\left(1,\, -\frac{g+c}{2h}\right) = a+b+g\left(-\frac{g+c}{2h}\right) + c\left(-\frac{g+c}{2h}\right) + h\left(-\frac{g+c}{2h}\right)^2 + k = a+b-\frac{(g+c)^2}{4h} + k = V_{\text{opt}} \end{aligned} \tag{61}$$

is reached on the set (47). It is seen from (48) and (49) that the roots of the corresponding equation (16) are $x_1 = \frac{g(g+c) - 2h(a+b)}{2ah}$ and $x_2 = 1$. But while $a+b+g = 0$ at $g+c+2h > 0$ then

$$\frac{g(g+c) - 2h(a+b)}{2ah} = \frac{g(g+c) + 2hg}{2ah} = \frac{g(g+c+2h)}{2ah} < 0 \tag{62}$$

and $x_1 \notin X$, $x_2 \in X$ and there is the set (50), whence the investigated subcase game solution is the set (51).

Subcase 3.2. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a+b+g = 0$; $g+c+2h = 0$. The point $y_{\min}^{\langle 1 \rangle} = -\frac{g+c}{2h} = 1$ and so the minimum of the function (60)

$$\begin{aligned} \min_{y \in Y} \max_{x \in X} S(x,\, y) &= \min\left\{ \min_{y \in [0;\, 1]} S(1,\, y),\, \min_{y \in \{1\}} S(0,\, y) \right\} = \min\left\{ S\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\, S(0,\, 1) \right\} = \min\left\{ S\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\, S(1,\, 1) \right\} = \\ &= S\left(1,\, y_{\min}^{\langle 1 \rangle}\right) = S(1,\, 1) = a+b+g+c+h+k = c+h+k = V_{\text{opt}} \end{aligned} \tag{63}$$

is reached on the set (23). The roots of the corresponding equation (16) are

$$x_1 = \frac{g(g+c) - 2h(a+b)}{2ah} = \frac{g(g+c) + 2hg}{2ah} = \frac{g(g+c+2h)}{2ah} = 0 \tag{64}$$

and $x_2 = 1$. Then here is the set (30) and in the being investigated subcase the probabilities $P(x_1) = P(0)$ and $P(x_2) = P(1)$ should be determined from the right inequality (31), where $y \neq y_{\text{opt}} = 1$, that is $\forall\ y < 1$:

$$V_{\text{opt}} = S(0,\,1) = S(1,\,1) = c + h + k \leqslant S(0,\,y)P(0) + S(1,\,y)P(1) =$$
$$= \left(cy + hy^2 + k\right)P(0) + \left(a + b + gy + cy + hy^2 + k\right)P(1) = cy + hy^2 + \left(a + b + gy\right)P(1) + k\,; \tag{65}$$
$$c + h - cy - hy^2 = c(1-y) + h(1-y)(1+y) =$$
$$= (1-y)\left[c + h(1+y)\right] \leqslant (a + b + gy)P(1) = (gy - g)P(1) = g(y-1)P(1)\,. \tag{66}$$

As $g(y-1) > 0 \;\; \forall\; y < 1$ then thereupon is an inequality for determining the probability $P(1) \;\; \forall\; y < 1$:

$$\frac{(1-y)\left[c + h(1+y)\right]}{g(y-1)} = -\frac{c + h + hy}{g} \leqslant P(1)\,. \tag{67}$$

However $hy < h$, $c + h + hy < c + 2h$ and $-\dfrac{c+h+hy}{g} < -\dfrac{c+2h}{g} = 1$. Thus $-\dfrac{c+h+hy}{g} < 1 \;\; \forall\; y < 1$ and inasmuch as

$$\lim_{\substack{y \in [0;\,1) \\ y \to 1}} \left(-\frac{c + h + hy}{g}\right) = -\frac{c + 2h}{g} = 1 \tag{68}$$

the probability $P(1)$ is

$$P(1) \in \lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left[-\frac{c + 2h}{g} - \varepsilon;\, 1\right] = \lim_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left[1 - \varepsilon;\, 1\right] = \{1\}\,. \tag{69}$$

Finally, the probabilities $P(1) = 1$ and $P(0) = 1 - P(1) = 0$, so in the investigated boundary subcase there is the set (50) and the game solution (59), that is

$$\mathscr{S} = \left\{\{1\},\, \{1\},\, c + h + k\right\}\,. \tag{70}$$

Subcase 3.3. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g = 0$; $g + c + 2h < 0$. Here the point $y_{\min}^{\langle 1 \rangle} = -\dfrac{g + c}{2h} > 1$ and there is the double parabolic inequality (56), which drives to the minimum of the function (60)

$$\min_{y \in Y} \max_{x \in X} S(x,\,y) = \min\left\{\min_{y \in [0;\,1]} S(1,\,y),\, \min_{y \in \{1\}} S(0,\,y)\right\} = \min\left\{\min\left\{S(1,\,0),\, S(1,\,1)\right\},\, S(1,\,1)\right\} =$$
$$= S(1,\,1) = a + b + g + c + h + k = c + h + k = V_{\text{opt}}\,, \tag{71}$$

that is reached on the set (23). The roots of the corresponding equation (16) are $x_1 = 0$ and $x_2 = 1$, so here again are true the statements (65) — (67). But by the initial condition $g + c + 2h < 0$ this subcase includes also the double inequality

$$-\frac{c + h + hy}{g} < -\frac{c + 2h}{g} < 1\,. \tag{72}$$

But there may be either $-\dfrac{c+2h}{g} \geqslant 0$ or $-\dfrac{c+2h}{g} < 0$, that is there are else two subcases with $c + 2h \geqslant 0$ and $c + 2h < 0$ respectively.

Subcase 3.3.1. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a + b + g = 0$; $g + c + 2h < 0$; $c + 2h \geqslant 0$. Here the value $-\dfrac{c+2h}{g} \geqslant 0$ and, subsequently,

$$\lim_{\substack{y \in [0;\,1) \\ y \to 1}} \left(-\frac{c + h + hy}{g}\right) = -\frac{c + 2h}{g} \in [0;\,1) \tag{73}$$

and here yet the probability $P(1)$ is

$$P(1) \in \lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \left[-\frac{c+2h}{g} - \varepsilon;\, 1 \right] = \left[-\frac{c+2h}{g};\, 1 \right]. \tag{74}$$

Finally, the probability

$$1 - P(1) = P(0) \in \left[0;\, \frac{g+c+2h}{g} \right], \tag{75}$$

where $\frac{g+c+2h}{g} \in (0;\, 1)$. Subsequently, the investigated boundary subcase has the set

$$\mathscr{X}_{\text{opt}} = \left\{ \{0,\, 1\},\, \{1 - P(1),\, P(1)\} \right\} \tag{76}$$

with the probability (74) and the game solution

$$\mathscr{S} = \left\{ \left\{ \{0,\, 1\},\, \{1 - P(1),\, P(1)\} \right\},\, \{1\},\, c+h+k \right\}. \tag{77}$$

Subcase 3.3.2. $a > 0$, $b > 0$, $g < 0$, $c < 0$; $a+b+g = 0$; $g+c+2h < 0$; $c+2h < 0$. Here the value $-\frac{c+2h}{g} < 0$ and, subsequently,

$$\lim_{\substack{y\in[0;\,1) \\ y\to 1}} \left(-\frac{c+h+hy}{g} \right) = -\frac{c+2h}{g} < 0 \tag{78}$$

and here yet the probability $P(1)$ is

$$P(1) \in \left\{ \lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \left[-\frac{c+2h}{g} - \varepsilon;\, 1 \right] \right\} \cap [0;\, 1] = \left[-\frac{c+2h}{g};\, 1 \right] \cap [0;\, 1] = [0;\, 1] = X. \tag{79}$$

Finally it is apparent, that the probability

$$1 - P(1) = P(0) \in [0;\, 1], \tag{80}$$

where the statement (79) is true. Subsequently, the investigated boundary subcase possesses the set (76) with the probability $P(1) \in [0;\, 1]$ in the statement (79), and the game solution is the set (77).

Conclusion

Having investigated the 12 subcases of the coefficients interrelationships in the kernel (3), there have been determined the corresponding eight types of the continuous antagonistic game solution. They are grouped in the table 1.

Table 1

The solutions of the investigated continuous strictly convex antagonistic game with the kernel

$S(x,\, y) = ax^2 + bx + gxy + cy + hy^2 + k$

The given game kernel attributes with $a > 0$, $b > 0$, $g < 0$, $c < 0$	The game solution $\mathscr{S} = \left\{ \mathscr{X}_{\text{opt}},\, \mathscr{Y}_{\text{opt}},\, V_{\text{opt}} \right\}$
$a+b+g < 0$, $g+c+2h < 0$, $2h(a+b) - cg < 0$, $c+2h \geqslant 0$	$\mathscr{S} = \left\{ \{0\},\, \left\{ -\frac{c}{2h} \right\},\, k - \frac{c^2}{4h} \right\}$

<table>
<tr><th>The given game kernel attributes with $a>0$, $b>0$, $g<0$, $c<0$</th><th>The game solution $\mathscr{S}=\left\{\mathscr{X}_{\text{opt}},\,\mathscr{Y}_{\text{opt}},\,V_{\text{opt}}\right\}$</th></tr>
<tr><td>$a+b+g<0$, $g+c+2h\geqslant 0$,
$2h(a+b)-g(g+c)\leqslant 0$,
$2h(a+b)-cg<0$</td><td></td></tr>
<tr><td>$a+b+g<0$, $g+c+2h<0$,
$2h(a+b)-cg<0$, $c+2h<0$</td><td>$\mathscr{S}=\left\{\{0\},\,\{1\},\,c+h+k\right\}$</td></tr>
<tr><td>$a+b+g<0$, $g+c+2h<0$,
$2h(a+b)-cg\geqslant 0$</td><td rowspan="2">$\mathscr{S}=\left\{\left\{\{0,\,1\},\,\{1-P(1),\,P(1)\}\right\},\,\left\{-\dfrac{a+b}{g}\right\},\,h\dfrac{(a+b)^2}{g^2}-c\dfrac{a+b}{g}+k\right\}$,
$P(1)=\dfrac{2h(a+b)-cg}{g^2}$</td></tr>
<tr><td>$a+b+g<0$, $g+c+2h\geqslant 0$,
$2h(a+b)-g(g+c)\leqslant 0$,
$2h(a+b)-cg\geqslant 0$</td></tr>
<tr><td>$a+b+g<0$, $g+c+2h\geqslant 0$,
$2h(a+b)-g(g+c)>0$</td><td rowspan="3">$\mathscr{S}=\left\{\{1\},\,\left\{-\dfrac{g+c}{2h}\right\},\,a+b-\dfrac{(g+c)^2}{4h}+k\right\}$</td></tr>
<tr><td>$a+b+g>0$, $g+c+2h\geqslant 0$</td></tr>
<tr><td>$a+b+g=0$, $g+c+2h>0$</td></tr>
<tr><td>$a+b+g>0$, $g+c+2h<0$</td><td>$\mathscr{S}=\left\{\{1\},\,\{1\},\,a+b+g+c+h+k\right\}$</td></tr>
<tr><td>$a+b+g=0$, $g+c+2h=0$</td><td>$\mathscr{S}=\left\{\{1\},\,\{1\},\,c+h+k\right\}$</td></tr>
<tr><td>$a+b+g=0$, $g+c+2h<0$, $c+2h\geqslant 0$</td><td>$\mathscr{S}=\left\{\left\{\{0,\,1\},\,\{1-P(1),\,P(1)\}\right\},\,\{1\},\,c+h+k\right\}$,
$P(1)\in\left[-\dfrac{c+2h}{g};\,1\right]$</td></tr>
<tr><td>$a+b+g=0$, $g+c+2h<0$, $c+2h<0$</td><td>$\mathscr{S}=\left\{\left\{\{0,\,1\},\,\{1-P(1),\,P(1)\}\right\},\,\{1\},\,c+h+k\right\}$,
$P(1)\in[0;\,1]$</td></tr>
</table>

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